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Short Communication

Discussion of the paper "Tensorial form definitions of discrete mechanical quantities for granular assemblies" [M. Satake, Int. J. Solids and Structures 2004, 41(21), pp. 5775–5791]

Katalin Bagi *

Research Group for Computational Structural Mechanics of the Hungarian Academy of Sciences, Department of Structural Mechanics, Budapest University of Technology and Economics, Muegyetem rkp. 3.K.mf.35., H-1521 Budapest, Hungary

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Abstract

The aim of this Discussion is to clarify a terminological issue in a previous IJSS paper.

Keywords: Cell system; Dirichlet; Laguerre; Voronoi; Radical plane tessellation

1. Introductory remarks

Satake (2004) proposes a definition for stress and strain for an assembly of unequal spheres with the help of contact cells, a system of polyhedra carrying the stresses as well as the deformations of the assembly according to the approach of the paper. The assemblies considered by Satake (Fig. 1) consist of spherical particles that may have contacts with each other. The spheres may intersect, but because of the mechanical background of the problem, these intersections are small compared to the sphere radii.

The definition of contact cells is based on a cell complex determined by the *power planes* of the spheres. (The power plane of two spheres is the set of those points having equal tangent lengths to the two spheres.) This cell complex is named 'Dirichlet tessellation' in my paper Bagi (1995), and in Satake (2004) citing my work.

Satake and myself were informed recently that the cell complex we called 'Dirichlet tessellation' for a collection of unequal spheres is known under other names in the mathematical literature. In order to clarify this terminological confusion, an overview will be given here on the history and on the different existing names of this and similar other tessellations. (The figures are in 2D, but most of them are illustrations of 3D systems as indicated in the text.)

* Fax: +36 1 4631099.
E-mail address: kbagi@mail.bme.hu

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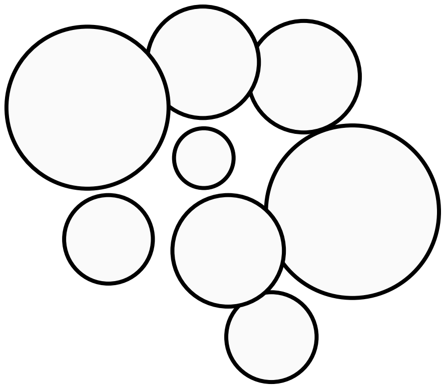

Fig. 1. 2D illustration of the considered assemblies of unequal spheres.

The basic subject of this discussion is the following problem:

Given a set of unequal spheres in the 3D or 2D Euclidean space. The spheres may have point-like contacts, and they may have intersections that are small. The space should be subdivided into cells in such a way that the following requirements have to be satisfied:

1. Every cell corresponds to exactly one sphere and vice versa.
2. The cell contains the center of the corresponding sphere.
3. Every point of the space belongs to exactly one cell, except from the common faces of the cells. (This requirement involves that the union of the cells completely covers the space, without any gaps.)

2. The generalized Dirichlet cell complex

In his classical paper Dirichlet (1850) considers a set of N points $O_1, O_2, \ldots, O_N$ in the Euclidean space. The space is divided into (convex) cells in such a way that any point P of the space is assigned to the O_i point closest to it (Fig. 2)

$$d^{PO_i} \leqslant d^{PO_l} \quad \text{for all } l \neq i.$$

The common face of the cells of O_j and O_k is planar. It is formed by those points Q whose distance from O_j and O_k are equal and not longer than the distance from any other O_l:

$$d^{PO_j} = d^{PO_k} \leqslant d^{PO_l} \quad \text{for all} \quad l \neq j \quad \text{and} \quad l \neq k.$$

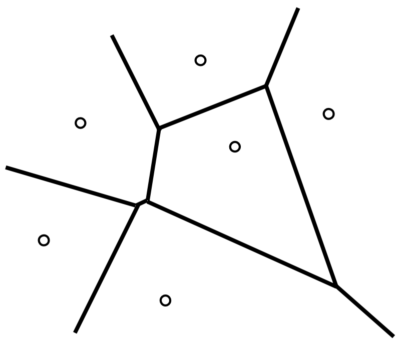

Fig. 2. The Dirichlet cell complex for a set of points.

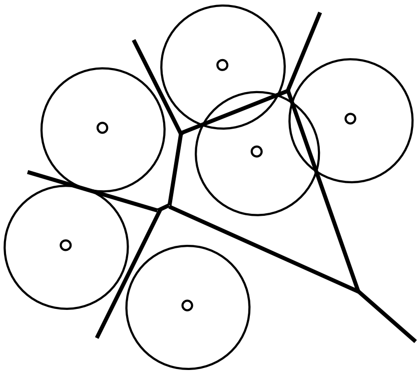

Fig. 3. The Dirichlet cell complex for the centers of equal spheres.

The definition is valid in 2D and 3D as well. The resulting cell complex covers the space without any gaps, and every point of the space belongs to exactly one cell, except from the common faces of neighboring cells.

The generalization of this system for arbitrary *m*-dimensional case was introduced by Voronoi (1908).

Instead of the points $O_1, O_2, \ldots, O_N$, consider now a set of equal circles/spheres $S_1, S_2, \ldots, S_N$ whose centers are $O_1, O_2, \ldots, O_N$, respectively. The definition of the Dirichlet cell complex can easily be applied for this case also. If every point P of the space is assigned to that sphere whose center is closest to P, the space will be divided into cells $C_1, C_2, \ldots, C_N$ that satisfy the above requirements (Fig. 3). (Note that center of sphere O_i belongs to the C_i cell of O_i. If two spheres O_j and O_k have a single contact point, then the contact point is on the common face of the corresponding cells, C_j and C_k, so the contact point belongs to both cells.)

The problem is, however, more difficult if the spheres have different radii. A generalization of the 2D Dirichlet cell complex was suggested by Fejes Toth (1953) for unequal circles, based on the *power lines* of neighboring (perhaps contacting) circles. The power line for a pair of circles is the set of those points having equal tangent length to the two circles (Fig. 4).

Fejes Toth (1953) deals with unequal circles in 2D that may have point-like contacts, and small intersections (small in the sense that the center of any circle cannot be the internal point of any other circle). The generalized Dirichlet cell complex of such a set of unequal circles can be defined in the following way (see Fig. 5 too):

Given a set of unequal circles $S_1, S_2, \ldots, S_N$ with their centers $O_1, O_2, \ldots, O_N$. In order to prepare cell C_i that belongs to circle S_i, determine the power line of S_i and another circle S_j. The power line divides the plane into

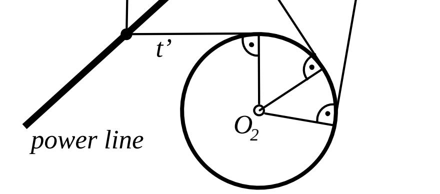

Fig. 4. The power line.

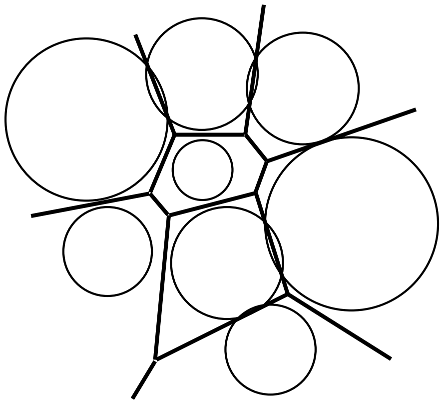

Fig. 5. The generalized Dirichlet cells of Fejes Toth.

two half-planes; consider now the half-plane containing O_i, together with the power line itself. Do the same for all $j \neq i$. The cell C_i is the common part of these half-planes.

Note that the common face of two neighboring cells is a part of the power line of the corresponding two circles. The common face belongs to both cells.

The generalized Dirichlet cell complex proposed by Fejes Toth satisfies the above requirements (1–3), and it can easily be extended to 3D where the power planes determine the cells that belong to the given spheres.

An important feature of the generalized Dirichlet cell complex is that all cells are convex and have straight faces. This is why Satake (2004) could advantageously apply the 3D extension of the generalized Dirichlet cell complex for the geometrical modeling of granular systems.

3. The Laguerre diagram

The mathematical literature is rich in definitions of tessellations satisfying the above requirements 1–3 (different possibilities can be found e.g. in Okabe et al. (2000)); and some of the proposed systems consist of convex cells with straight/planar faces even in the case of unequal circles/spheres. Imai et al. (1985), for instance, consider a set of unequal circles in the 2D Euclidean space. The circles may intersect, and they may even be embedded into each other. The concept of 'Laguerre distance' is applied for the definition of their cell complex: the Laguerre distance of a point P (located at x_1^P, x_2^P) from a circle S_i with center O_i (located at x_1^i, x_2^i) and radius R_i is

$$\mathrm{pow}_{P-i} = (x_1^P - x_1^i)^2 + (x_2^P - x_2^i)^2 - R_i^2. \tag{1}$$

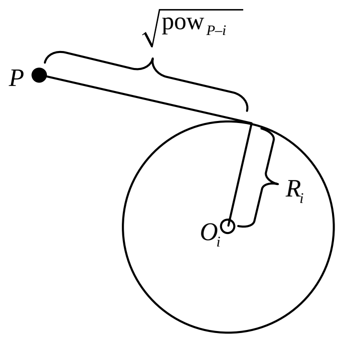

Fig. 6. The meaning of pow_{P-i} if P is outside S_i.

Note that if P is outside S_i, then pow_{P-i} is the square length of the tangent section from P to S_i (see Fig. 6); and pow_{P-i} can be negative, iff P is inside S_i. (The concept of Laguerre distance is explained in more details in Okabe et al. (2000).)

A point P belongs to the cell of the circle of O_i iff

$$\mathrm{pow}_{P-i} \leqslant \mathrm{pow}_{P-j} \quad \text{for all } j \neq i. \tag{2}$$

The cell complex formed this way is called 'the Voronoi diagram in the Laguerre geometry' in Imai et al. (1985), or shortly 'Laguerre diagram' in Okabe et al. (2000).

Gellatly and Finney (1982a,b) introduce the 'radical plane tessellation', a cell complex that is the 3D version of the Laguerre diagram. (They use this tessellation for the geometrical modeling of systems that consist of different kinds of atoms. The atoms are modeled by spheres whose radii are different for the different types of atoms. Because of the physical background, the authors do not consider the case of embedded spheres.)

Note that the radical plane tessellation of Gellatly and Finney is the same as the tessellation applied by Satake (2004).

Aurenhammer (1987) proposes a cell complex that can be considered as a generalization of the 2D definition of Imai et al. (1985) for d dimensions. He uses 'power diagram', 'Dirichlet cell complex' and 'Laguerre diagram' as synonymous names for his cell complex. There is, however, a slight difference between the approach of Aurenhammer and Imai et al.: the cells of Aurenhammer are, by definition, open sets, as they do not contain the power planes themselves (Aurenhammer uses a strict inequality instead of the $\leqslant$sign in (2)).

4. Conclusion

Satake (2004) applies the 3D version of the generalized Dirichlet cell complex defined by Fejes Toth (1953), a cell complex which is equivalent to the Laguerre diagram or to the radical plane tessellation of such an assembly where the intersections of the spheres are small. The cells are convex and they have planar faces, what makes them particularly advantageous for the analysis of granular assemblies.

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